

On Symmetric Polynomials

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Abstract

In this paper, we study structure theorems of algebras of symmetric functions. Based on a certain relation on elementary symmetric polynomials generating such algebras, we consider perturbation in the algebras. In particular, we understand generators of the algebras as perturbations. From such perturbations, define injective maps on generators, which induce algebra-monomorphisms (or embeddings) on the algebras. They provide inductive structure theorems on algebras of symmetric polynomials. As application, we give a computer algorithm, written in JAVA v. 8, for finding quantities from elementary symmetric polynomials.

Symmetric Functions, Elementary Symmetric Polynomials, Symmetric Subalgebras, Perturbations.

05A19, 30E50, 37E99, 44A60.

1 Introduction

In this paper, we study structure theorems of algebras generated by symmetric polynomials with commutative multi-variables. By establishing certain recurrence relations on symmetric polynomials, we prove our structure theorems. As application, we consider how to construct a degree- $(n+1)$ single-variable polynomial $f_{t_0}(z)$ from a given degree- n polynomial $f(z)$ by adding a zero t_0 , and we provide a computer algorithm, written in the computer language JAVA version 8; for finding quantities obtained from elementary symmetric polynomials.

Throughout this paper, fix $n \in \mathbb{N}$, with additional condition: $n > 1$, and let x_1, \dots, x_n be arbitrary commutative *variables* (or indeterminants), for $n \in \mathbb{N}$. Then one can have an algebra

(1.1)

$$\mathcal{A}_{x_1, \dots, x_n} = \mathbb{C}[\{x_1, \dots, x_n\}],$$

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consisting of all n -variable polynomials in x_1, \dots, x_n . We call $\mathcal{A}_{x_1, \dots, x_n}$ of (1.1), the n -variable polynomial algebra.

i.e., if $f \in \mathcal{A}_{x_1, \dots, x_n}$, then it is expressed by

$$f = f(x_1, \dots, x_n) = t_0 + \sum_{j=1}^k \sum_{(r_1, \dots, r_j) \in \{1, \dots, n\}^j} t_{(r_1, \dots, r_j)} \left(\prod_{i=1}^j x_{r_i} \right),$$

with $t_0, t_{(r_1, \dots, r_j)} \in \mathbb{C}$, for $k \in \mathbb{N}$.

Let X be a finite set. The symmetric group S_X on X is a group under the usual functional composition consisting of all bijective maps, called permutations, on X . If $X = \{1, \dots, n\}$, then we denote S_X simply by S_n , for $n \in \mathbb{N}$.

Definition 1.1 An element f of $\mathcal{A}_{x_1, \dots, x_n}$ of (1.1) is said to be symmetric, if

$$f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}),$$

for all $\sigma \in S_n$, where S_n is the symmetric group on $\{1, \dots, n\}$.

Define now the subset $\mathcal{S}_{x_1, \dots, x_n}$ of $\mathcal{A}_{x_1, \dots, x_n}$ by

(1.2)

$$\mathcal{S}_{x_1, \dots, x_n} = \{f \in \mathcal{A}_{x_1, \dots, x_n} : f \text{ is symmetric}\}.$$

We call $\mathcal{S}_{x_1, \dots, x_n}$, the symmetric subalgebra of $\mathcal{A}_{x_1, \dots, x_n}$.

It is not difficult to check that,

$$(1.3) \quad f_1, f_2 \in \mathcal{S}_{x_1, \dots, x_n} \implies f_1 + f_2 \in \mathcal{S}_{x_1, \dots, x_n},$$

$$(1.4) \quad f \in \mathcal{S}_{x_1, \dots, x_n} \text{ and } t \in \mathbb{C} \implies tf \in \mathcal{S}_{x_1, \dots, x_n},$$

and hence, the subset $\mathcal{S}_{x_1, \dots, x_n}$ of (1.2) forms a well-defined vector subspace of $\mathcal{A}_{x_1, \dots, x_n}$ over \mathbb{C} , by (1.3) and (1.4).

Moreover, one has

$$(1.5) \quad f_1, f_2 \in \mathcal{S}_{x_1, \dots, x_n} \implies f_1 f_2 \in \mathcal{S}_{x_1, \dots, x_n},$$

where $f_1 f_2$ means the usual functional multiplication of f_1 and f_2 in $\mathcal{A}_{x_1, \dots, x_n}$. Since

$$\begin{aligned} f_1(f_2 f_3) &= (f_1 f_2) f_3 \in \mathcal{S}_{x_1, \dots, x_n}, \\ f_1(f_2 + f_3) &= f_1 f_2 + f_1 f_3 \in \mathcal{S}_{x_1, \dots, x_n}, \end{aligned}$$

and

$$(f_1 + f_2) f_3 = f_1 f_3 + f_2 f_3 \in \mathcal{S}_{x_1, \dots, x_n},$$

by (1.5), whenever $f_1, f_2, f_3 \in \mathcal{S}_{x_1, \dots, x_n}$, the subspace $\mathcal{S}_{x_1, \dots, x_n}$ indeed forms a well-determined subalgebra of $\mathcal{A}_{x_1, \dots, x_n}$.

Let's define the following functions

(1.6)

$$\varepsilon_k(x_1, \dots, x_n) = \sum_{i_1 < i_2 < \dots < i_k \in \{1, \dots, n\}} \left(\prod_{l=1}^k x_{i_l} \right)$$

in $\mathcal{A}_{x_1, \dots, x_n}$, for all $k = 1, \dots, n$. Then they are symmetric in $\mathcal{A}_{x_1, \dots, x_n}$. i.e.,

$$\begin{aligned} \varepsilon_1(x_1, \dots, x_n) &= \sum_{j=1}^n x_j, \\ \varepsilon_2(x_1, \dots, x_n) &= \sum_{i_1 < i_2 \in \{1, \dots, n\}} x_{i_1} x_{i_2}, \\ \varepsilon_3(x_1, \dots, x_n) &= \sum_{i_1 < i_2 < i_3 \in \{1, \dots, n\}} x_{i_1} x_{i_2} x_{i_3}, \\ &\dots, \\ \varepsilon_n(x_1, \dots, x_n) &= \prod_{j=1}^n x_j \end{aligned}$$

are elements of $\mathcal{S}_{x_1, \dots, x_n}$.

Definition 1.2 We call such polynomials $\varepsilon_k(x_1, \dots, x_n)$ of (1.6), the k -th elementary symmetric polynomials of $\mathcal{S}_{x_1, \dots, x_n}$, for all $k = 1, \dots, n$.

Notation 1.1 In the rest of this paper, we denote $\varepsilon_k(x_1, \dots, x_n)$ simply by $\varepsilon_k^{1, \dots, n}$, for all $k = 1, \dots, n$. Also, for convenience, define

$$\varepsilon_0^{1, \dots, n} = 1 \text{ and } \varepsilon_{n+i}^{1, \dots, n} = 0,$$

as constant functions in $\mathcal{S}_{x_1, \dots, x_n}$, for all $i \in \mathbb{N}$. Whenever we want to emphasize the variables of $\varepsilon_k^{1, \dots, n}$ precisely, we denote them by $\varepsilon_k^{x_1, \dots, x_n}$, for $k = 1, \dots, n$. \square

The following proposition is well-known under its name: *Fundamental Theorem of Symmetric Functions*.

Proposition 1.1 (See [3]) Let $\mathcal{S}_{x_1, \dots, x_n}$ be the symmetric subalgebra (1.2) of the n -variable polynomial algebra $\mathcal{A}_{x_1, \dots, x_n}$ of (1.1). Then

$$\mathcal{S}_{x_1, \dots, x_n} \stackrel{Alg}{=} \mathbb{C} \left[\{\varepsilon_1^{1, \dots, n}, \varepsilon_2^{1, \dots, n}, \dots, \varepsilon_n^{1, \dots, n}\} \right], \quad (1.7)$$

where " $\stackrel{Alg}{=}$ " means "being algebra-isomorphic," and where $\varepsilon_k^{1, \dots, n}$ are the elementary symmetric polynomials in the sense of (1.6), for all $k = 1, \dots, n$. \square

The above structure theorem (1.7) shows that all symmetric functions in $\mathcal{A}_{x_1, \dots, x_n}$ are generated by the elementary symmetric polynomials $\{\varepsilon_k^{1, \dots, n}\}_{k=1}^n$ of (1.6).

For more about symmetric functions and related studies in mathematics, see e.g., [2], [3], and cited papers therein.

2 A Certain Relation on $\mathcal{S}_{x_1, \dots, x_n}$

In this section, we establish a recurrence relation on the elementary symmetric polynomials $\{\varepsilon_k^{1, \dots, n}\}_{k=1}^n$ generating the symmetric subalgebra $\mathcal{S}_{x_1, \dots, x_n}$ of the n -variable polynomial algebra $\mathcal{A}_{x_1, \dots, x_n}$.

As in **Notation 1.1**, for any $k \in \{1, \dots, n\}$,

$$\varepsilon_k^{1, \dots, n} = \varepsilon_k(x_1, \dots, x_n).$$

So, if we write $\varepsilon_k^{i_1, \dots, i_t}$, for $i_j \in \{1, \dots, n\}$, $j = 1, \dots, t$, with $t \leq n$ in \mathbb{N} , then it means

(2.1)

$$\varepsilon_k^{i_1, \dots, i_t} = \begin{cases} \varepsilon_k(x_{i_1}, \dots, x_{i_t}) & \text{if } k \leq t \\ 0 & \text{if } k > t, \end{cases}$$

Moreover, nonzero new elementary symmetric polynomials

$$\varepsilon_1^{i_1, \dots, i_t}, \dots, \varepsilon_t^{i_1, \dots, i_t}$$

are understood as the symmetric functions generating $\mathcal{S}_{x_{i_1}, \dots, x_{i_t}}$ in $\mathcal{A}_{x_{i_1}, \dots, x_{i_t}}$.

For example, if we write $\varepsilon_k^{1, \dots, n-1}$, for $k = 1, \dots, n-1$, then they are the elementary symmetric polynomials generating $\mathcal{S}_{x_1, \dots, x_{n-1}}$ in $\mathcal{A}_{x_1, \dots, x_{n-1}}$; if we write ε_1^t , then it is a single-variable function x_t in $\mathcal{A}_{x_t} = \mathbb{C}[x_t]$.

Theorem 2.1 *Let $\varepsilon_k^{1, \dots, n}$ be the elementary symmetric polynomials generating the symmetric subalgebra $\mathcal{S}_{x_1, \dots, x_n}$. Then*

(2.2)

$$\varepsilon_k^{1, \dots, n} = \varepsilon_k^{1, \dots, n-1} + \varepsilon_{k-1}^{1, \dots, n-1} \varepsilon_1^n,$$

where the summands and the factors of the right-hand side of (2.2) are in the sense of (2.1).

PROOF: Assume first that $k = 1$. Then

$$\begin{aligned} \varepsilon_1^{1, \dots, n} &= \sum_{j=1}^n x_j = \left(\sum_{j=1}^{n-1} x_j \right) + x_n \\ &= \varepsilon_1^{1, \dots, n-1} + 1 \cdot \varepsilon_1^n \\ &= \varepsilon_1^{1, \dots, n-1} + \varepsilon_0^{1, \dots, n-1} \varepsilon_1^n, \end{aligned}$$

by **Notation 1.1** and (2.1). So, if $k = 1$, then the relation (2.2) holds.

Suppose now that $k = n$. Then

$$\begin{aligned} \varepsilon_n^{1, \dots, n} &= \prod_{j=1}^n x_j = 0 + \left(\prod_{j=1}^{n-1} x_j \right) (x_n) \\ &= \varepsilon_n^{1, \dots, n-1} + \varepsilon_{n-1}^{1, \dots, n-1} \varepsilon_1^n, \end{aligned}$$

by **Notation 1.1** and (2.1). Thus, if $k = n$, then the relation (2.2) holds. Now, take $k \in \{2, \dots, n-1\}$, and define a set

$$T_k^{1,\dots,n} = \{(i_1, \dots, i_k) \in \{1, \dots, n\}^k : i_1 < i_2 < \dots < i_k\}.$$

Define also a set

$$T_k^{1,\dots,n-1} = \{(j_1, \dots, j_k) \in \{1, \dots, n-1\}^k : j_1 < \dots < j_k\},$$

and similarly,

$$T_{k-1}^{1,\dots,n-1} = \{(l_1, \dots, l_{k-1}) \in \{1, \dots, n-1\}^{k-1} : l_1 < \dots < l_{k-1}\}.$$

By the very construction, two sets $T_k^{1,\dots,n-1}$ and $T_{k-1}^{1,\dots,n-1}$ are understood as subsets of $T_k^{1,\dots,n}$, for all $k = 2, 3, \dots, n-1$.

Depending on the above sets, construct

$$X_k^{1,\dots,n} = \left\{ \prod_{l=1}^k x_{i_l} : (i_1, \dots, i_k) \in T_k^{1,\dots,n} \right\},$$

$$X_k^{1,\dots,n-1} = \left\{ \prod_{l=1}^k x_{i_l} : (i_1, \dots, i_k) \in T_k^{1,\dots,n-1} \right\},$$

and

$$X_{k-1}^{1,\dots,n-1} = \left\{ \prod_{l=1}^{k-1} x_{i_l} : (i_1, \dots, i_{k-1}) \in T_{k-1}^{1,\dots,n-1} \right\},$$

respectively.

Now, let's define

$$Y_{k-1}^{1,\dots,n-1} = \left\{ \left(\prod_{l=1}^{k-1} x_{i_l} \right) x_n : \prod_{l=1}^{k-1} x_{i_l} \in X_{k-1}^{1,\dots,n-1} \right\},$$

i.e.,

$$Y_{k-1}^{1,\dots,n-1} = X_{k-1}^{1,\dots,n-1} x_n,$$

where $Xx = \{yx : y \in X\}$. Notice that if $k = 2, \dots, n-1$, then

$$X_k^{1,\dots,n} = X_k^{1,\dots,n-1} \sqcup Y_{k-1}^{1,\dots,n-1},$$

set-theoretically, by the very definitions, where \sqcup means the disjoint union. Observe now that, whenever $k = 2, \dots, n-1$, we have

$$\begin{aligned}
\varepsilon_k^{1, \dots, n} &= \varepsilon_k(x_1, \dots, x_n) = \sum_{(x_{i_1}, \dots, x_{i_k}) \in X_k^{1, \dots, n}} \left(\prod_{l=1}^k x_{i_l} \right) \\
&= \sum_{(x_{i_1}, \dots, x_{i_k}) \in X_k^{1, \dots, n-1} \sqcup Y_{k-1}^{1, \dots, n-1}} \left(\prod_{l=1}^k x_{i_l} \right) \\
&\text{by (2.3)} \\
&= \sum_{(x_{i_1}, \dots, x_{i_k}) \in X_k^{1, \dots, n-1}} \left(\prod_{l=1}^k x_{i_l} \right) \\
&\quad + \sum_{(x_{i_1}, \dots, x_{i_{k-1}}, x_n) \in Y_{k-1}^{1, \dots, n-1}} \left(\left(\prod_{l=1}^{k-1} x_{i_l} \right) (x_n) \right) \\
&= \sum_{(x_{i_1}, \dots, x_{i_k}) \in X_k^{1, \dots, n-1}} \left(\prod_{l=1}^k x_{i_l} \right) \\
&\quad + \left(\sum_{(x_{i_1}, \dots, x_{i_{k-1}}, x_n) \in Y_{k-1}^{1, \dots, n-1}} \left(\prod_{l=1}^{k-1} x_{i_l} \right) \right) (x_n) \\
&= \varepsilon_k^{1, \dots, n-1} + \varepsilon_{k-1}^{1, \dots, n-1} \varepsilon_1^n.
\end{aligned}$$

Therefore, the relations (2.2) hold, for all $k = 2, \dots, n-1$.

So, the relations (2.2) hold, for all $k = 1, 2, \dots, n$.

QED

The above relation (2.2) shows the relations between the generators

$$\{\varepsilon_1^{1, \dots, n}, \dots, \varepsilon_n^{1, \dots, n}\} \text{ of } \mathcal{S}_{x_1, \dots, x_n}$$

and those

$$\{\varepsilon_1^{1, \dots, n-1}, \dots, \varepsilon_{n-1}^{1, \dots, n-1}\} \text{ of } \mathcal{S}_{x_1, \dots, x_{n-1}} \text{ and } \{\varepsilon_1^n\} \text{ of } \mathcal{S}_{x_n} = \mathcal{A}_{x_n}$$

Also, it shows that all generators of $\mathcal{S}_{x_1, \dots, x_n}$ are induced by the generators of

$$\mathcal{S}_{x_1, \dots, x_{n-1}} \text{ and } \mathcal{S}_{x_n},$$

by (2.1).

Motivated by (2.2), one can obtain the following generalized result.

Theorem 2.2 *Let $\{\varepsilon_k^{1, \dots, n}\}_{k=1}^n$ be the elementary symmetric polynomials generating the symmetric subalgebra $\mathcal{S}_{x_1, \dots, x_n}$. Then, for any fixed $i_0 \in \{1, \dots, n\}$, we have*

$$(2.4)$$

$$\varepsilon_k^{1,\dots,n} = \varepsilon_k^{1,\dots,i_0-1, i_0+1,\dots,n} + \varepsilon_{k-1}^{1,\dots,i_0-1,i_0+1,\dots,n} \varepsilon_1^{i_0},$$

for all $k = 1, 2, \dots, n$, where $\varepsilon_l^{i_1,\dots,i_m}$ are in the sense of (2.1).

PROOF: The proof of (2.4) is similarly done by that of (2.3), by replacing n to i_0 . QED

So, one can realize that the generators of symmetric subalgebra $\mathcal{S}_{x_1,\dots,x_n}$ are induced by the generators of

$$\mathcal{S}_{x_1,\dots,x_{i_0-1},x_{i_0+1},\dots,n} \text{ and } \mathcal{S}_{x_{i_0}} = \mathbb{C}[x_{i_0} = \varepsilon_1^{i_0}],$$

by (2.4).

Example 2.1 Suppose we have the symmetric subalgebra $\mathcal{S}_{x_1,x_2,x_3,x_4}$ whose generators are the elementary symmetric polynomials $\varepsilon_j^{1,\dots,4}$, for $j = 1, \dots, 4$. i.e.,

$$\begin{aligned} \varepsilon_1^{1,2,3,4} &= x_1 + x_2 + x_3 + x_4, \\ \varepsilon_2^{1,2,3,4} &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4, \\ \varepsilon_3^{1,2,3,4} &= x_1x_2x_3 + x_1x_3x_4 + x_2x_3x_4, \end{aligned}$$

and

$$\varepsilon_4^{1,2,3,4} = x_1x_2x_3x_4$$

in $\mathcal{S}_{x_1,x_2,x_3,x_4}$.

Then

$$\begin{aligned} \varepsilon_1^{1,2,3,4} &= \varepsilon_1^{1,2,3} + \varepsilon_0^{1,2,3} \varepsilon_1^4 = \varepsilon_1^{1,2,3} + \varepsilon_1^4, \\ \varepsilon_2^{1,2,3,4} &= \varepsilon_2^{1,2,3} + \varepsilon_1^{1,2,3} \varepsilon_1^4, \\ \varepsilon_3^{1,2,3,4} &= \varepsilon_3^{1,2,3} + \varepsilon_2^{1,2,3} \varepsilon_1^4, \end{aligned}$$

and

$$\varepsilon_4^{1,2,3,4} = \varepsilon_4^{1,2,3} + \varepsilon_3^{1,2,3} \varepsilon_1^4 = \varepsilon_3^{1,2,3} \varepsilon_1^4.$$

Similarly, one has

$$\begin{aligned} \varepsilon_1^{1,2,3,4} &= \varepsilon_1^{1,2,4} + \varepsilon_0^{1,2,4} \varepsilon_1^3 = \varepsilon_1^{1,2,4} + \varepsilon_1^3, \\ \varepsilon_2^{1,2,3,4} &= \varepsilon_2^{1,2,4} + \varepsilon_1^{1,2,4} \varepsilon_1^3, \\ \varepsilon_3^{1,2,3,4} &= \varepsilon_3^{1,2,4} + \varepsilon_2^{1,2,4} \varepsilon_1^3, \end{aligned}$$

and

$$\varepsilon_4^{1,2,3,4} = \varepsilon_4^{1,2,4} + \varepsilon_3^{1,2,4} \varepsilon_1^3 = \varepsilon_3^{1,2,4} \varepsilon_1^3,$$

etc.

Acknowledgment Before submitting this paper, the authors realized that the relation (2.2) was already known in psychology; psychological tests (e.g, see [1]). According to [1], Dr. Fischer proved the relation (2.2) in his book "Einführung in die Theorie Psychologischer Tests." However, we could not find the sources, including the book, containing Fischer's proof. So, we provided our own proofs of (2.2) and (2.4) above. \square

3 Perturbations on \mathfrak{S} and Shifts on $\mathcal{S}_{x_1, \dots, x_n}$

In this section, we define the collection \mathfrak{S} of all symmetric subalgebras of finitely-many commutative variables. And construct certain perturbation processes on \mathfrak{S} , by understanding each f of $\mathcal{S}_{x_1, \dots, x_n} \in \mathfrak{S}$ as multiplication from $\mathcal{S}_{y_1, \dots, y_k}$ to $\mathcal{S}_{x_1, \dots, x_n, y_1, \dots, y_k}$ in \mathfrak{S} under additional axiomatization (See Section 3.1). Also, we consider a shifting process on a fixed symmetric subalgebra $\mathcal{S}_{x_1, \dots, x_n} \in \mathfrak{S}$ by reformulating indexes of generators of $\mathcal{S}_{x_1, \dots, x_n}$, in Section 3.2. In Section 3.3, we apply our perturbation of Section 3.1 and shifting process of Section 3.2 to the inductive construction processes of \mathfrak{S} .

3.1 Perturbation on \mathfrak{S}

Let \mathfrak{S} be the collection of symmetric subalgebras $\mathcal{S}_{x_1, \dots, x_n}$ in commutative variables $\{x_1, \dots, x_n\}$, for all $n \in \mathbb{N}$. Let's fix $n_0 \in \mathbb{N}$, and take $\mathcal{S}_{x_1, \dots, x_{n_0}}$ in \mathfrak{S} .

Definition 3.1 Define now perturbations of $\mathcal{S}_{x_1, \dots, x_{n_0}}$ on \mathfrak{S} by
(3.1.1)

$$f : h \in \mathcal{S}_{y_1, \dots, y_n} \mapsto fh \in \mathcal{S}_{x_1, \dots, x_{n_0}, y_1, \dots, y_n} \text{ in } \mathfrak{S},$$

for $f \in \mathcal{S}_{x_1, \dots, x_{n_0}}$, for all $\mathcal{S}_{y_1, \dots, y_n} \in \mathfrak{S}$, with identification: if

$$\partial = \{x_1, \dots, x_{n_0}\} \cap \{y_1, \dots, y_n\}$$

is non-empty, then

(3.1.1)'

$$\mathcal{S}_{x_1, \dots, x_{n_0}, y_1, \dots, y_n} = \mathcal{S}_{(\{x_1, \dots, x_{n_0}\} \setminus \partial) \sqcup \partial \sqcup (\{y_1, \dots, y_n\} \setminus \partial)},$$

in \mathfrak{S} .

Since every symmetric subalgebra is generated by elementary symmetric polynomials, the perturbations (3.1.1) of $\mathcal{S}_{x_1, \dots, x_{n_0}}$ on \mathfrak{S} is characterized on generators, i.e.,

(3.1.2)

$$\varepsilon_k^{x_1, \dots, x_{n_0}} : \varepsilon_l^{y_1, \dots, y_n} \in \mathcal{S}_{y_1, \dots, y_n} \mapsto \varepsilon_k^{x_1, \dots, x_{n_0}} \varepsilon_l^{y_1, \dots, y_n} \in \mathcal{S}_{x_1, \dots, x_{n_0}, y_1, \dots, y_n}$$

in \mathfrak{S} , satisfying the identification; (3.1.1)'.

Now, let $\mathcal{S}_y = \mathbb{C}[y] = \mathcal{A}_y$ in \mathfrak{S} , and assume that $y \neq x_j$, for $j = 1, \dots, n_0$. Consider the perturbation of $\mathcal{S}_{x_1, \dots, x_{n_0}}$ on \mathfrak{S} acting on \mathcal{S}_y , i.e.,

$$\varepsilon_k^{x_1, \dots, x_{n_0}} : \varepsilon_1^y = y \mapsto \varepsilon_k^{x_1, \dots, x_{n_0}} \varepsilon_1^y.$$

Proposition 3.1 The perturbations of $\mathcal{S}_{x_1, \dots, x_{n_0}}$ on \mathfrak{S} is a well-defined categorical functor on \mathfrak{S} .

PROOF: The proof is from the very definition (3.1.1), with identification (3.1.1)', with help of (3.1.2). QED

3.2 Shifts on $\mathcal{S}_{x_1, \dots, x_n}$

Let $\mathcal{S}_{x_1, \dots, x_n} \in \mathfrak{S}$ be a symmetric subalgebra. Define a shift U on $\mathcal{S}_{x_1, \dots, x_n}$ by a linear multiplicative transformation on $\mathcal{S}_{x_1, \dots, x_n}$ satisfying

(3.2.1)

$$U : \varepsilon_k^{1, \dots, n} \mapsto \varepsilon_{k-1}^{1, \dots, n} \text{ on } \mathcal{S}_{x_1, \dots, x_n},$$

for all $k = 1, \dots, n$, with additional axiomatization;

(3.2.1)'

$$U(\varepsilon_0^{1, \dots, n}) = U(1) = 0,$$

making all constant functions of \mathbb{C} in $\mathcal{S}_{x_1, \dots, x_n}$ be zero, i.e., $U(C) = 0$, for all $C \in \mathbb{C}$ in $\mathcal{S}_{x_1, \dots, x_n}$.

Definition 3.2 *The morphism U on $\mathcal{S}_{x_1, \dots, x_n}$ of (3.2.1) is called the shift on $\mathcal{S}_{x_1, \dots, x_n}$.*

More generally, the shift U on $\mathcal{S}_{x_1, \dots, x_n}$ satisfies

$$\begin{aligned} (3.2.2) \quad & U \left(t_0 + \sum_{t=0}^k \sum_{(i_1, \dots, i_k) \in \{1, \dots, n\}^k} t_{i_1, \dots, i_k} \prod_{j=1}^k \varepsilon_{i_j}^{1, \dots, n} \right) \\ &= \sum_{t=0}^k \sum_{(i_1, \dots, i_k) \in \{1, \dots, n\}^k} t_{i_1, \dots, i_k} \prod_{j=1}^k \varepsilon_{i_j-1}^{1, \dots, n}, \end{aligned}$$

under (3.2.1)', where $t_{i_1, \dots, i_k} \in \mathbb{C}$.

Proposition 3.2 *The shift U of (3.2.1), satisfying (3.2.2), is a well-defined algebra-homomorphism on $\mathcal{S}_{x_1, \dots, x_n}$, for all $\mathcal{S}_{x_1, \dots, x_n} \in \mathfrak{S}$.*

PROOF: Let U be the shift (3.1.1) on $\mathcal{S}_{x_1, \dots, x_n} \in \mathfrak{S}$ satisfying (3.1.1)'. By the very construction, the morphism U is a linear transformation which is multiplicative. So, it is automatically an algebra-homomorphism.

Clearly, by the linearity, one has

$$U(t_1 f_1 + t_2 f_2) = t_1 U(f_1) + t_2 U(f_2),$$

for all $t_1, t_2 \in \mathbb{C}$ and $f_1, f_2 \in \mathcal{S}_{x_1, \dots, x_n}$.

Also, by the multiplicativity of U ,

$$\begin{aligned} & U \left(\left(\varepsilon_{k_1}^{1, \dots, n} \right)^{m_1} \dots \left(\varepsilon_{k_l}^{1, \dots, n} \right)^{m_l} \right) = U \left(\left(\varepsilon_{k_1}^{1, \dots, n} \right)^{m_1} \right) \dots U \left(\left(\varepsilon_{k_l}^{1, \dots, n} \right)^{m_l} \right) \\ &= \left(U \left(\varepsilon_{k_1}^{1, \dots, n} \right) \right)^{m_1} \dots \left(U \left(\varepsilon_{k_l}^{1, \dots, n} \right) \right)^{m_l} \\ &= \left(\varepsilon_{k_1-1}^{1, \dots, n} \right)^{m_1} \dots \left(\varepsilon_{k_l-1}^{1, \dots, n} \right)^{m_l}, \end{aligned}$$

by (3.2.2), for all $k_1, \dots, k_l = 0, 1, \dots, n$ (with (3.2.1)'), for all $m_1, \dots, m_l \in \mathbb{N}$, for all $l \in \mathbb{N}$.

So, for any $f_1, f_2 \in \mathcal{S}_{x_1, \dots, x_n}$, we have

$$U(f_1 f_2) = U(f_1) U(f_2) \text{ in } \mathcal{S}_{x_1, \dots, x_n}.$$

i.e., the morphism U is indeed an algebra-homomorphism on $\mathcal{S}_{x_1, \dots, x_n}$. QED

3.3 From $\mathcal{S}_{x_1, \dots, x_n}$ to $\mathcal{S}_{x_1, \dots, x_n, y}$ in \mathfrak{S}

Now, let's fix $n_0 \in \mathbb{N}$, and the symmetric subalgebra $\mathcal{S}_{x_1, \dots, x_{n_0}}$ in \mathfrak{S} . Also, fix a symmetric subalgebra $\mathcal{S}_y = \mathbb{C}[y] = \mathcal{A}_y$ in \mathfrak{S} , where $y \neq x_j$, for all $j = 1, \dots, n_0$. Let

$$\mathcal{E}_{x_1, \dots, x_{n_0}} = \{\varepsilon_k^{x_1, \dots, x_{n_0}}\}_{k=1}^{n_0}$$

be the generator set of $\mathcal{S}_{x_1, \dots, x_{n_0}}$. If we understand generators of $\mathcal{E}_{x_1, \dots, x_{n_0}}$ as the perturbations (3.1.1) on \mathfrak{S} , then they act

$$\varepsilon_k^{x_1, \dots, x_{n_0}}(\varepsilon_1^y) = \varepsilon_k^{x_1, \dots, x_{n_0}} \varepsilon_1^y \in \mathcal{S}_{x_1, \dots, x_{n_0}, y}$$

on \mathcal{S}_y , for all $k = 0, 1, \dots, n_0$, with identity: $\varepsilon_0^{x_1, \dots, x_{n_0}} = 1$.

On the perturbations $\mathcal{E}_{x_1, \dots, x_{n_0}}$ of (3.3.1), consider

$$U(\varepsilon_k^{x_1, \dots, x_{n_0}}) = \varepsilon_{k-1}^{x_1, \dots, x_{n_0}}, \text{ for all } k = 1, \dots, n_0,$$

where U is the shift on $\mathcal{S}_{x_1, \dots, x_{n_0}}$ of (3.2.1) satisfying (3.2.1)'.

Define now a morphism

$$\alpha : \mathcal{E}_{x_1, \dots, x_{n_0}} \rightarrow \mathcal{S}_{x_1, \dots, x_{n_0}, y},$$

by a function satisfying

(3.3.3)

$$\alpha(\varepsilon_k^{x_1, \dots, x_{n_0}}) = \varepsilon_k^{x_1, \dots, x_{n_0}} + U(\varepsilon_k^{x_1, \dots, x_{n_0}}) \varepsilon_1^y.$$

Theorem 3.3 *The function α of (3.3.3) is a well-defined injective function from $\mathcal{E}_{x_1, \dots, x_{n_0}}$ into $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$. Furthermore, this function α of (3.3.3) is injective from the generator set $\mathcal{E}_{x_1, \dots, x_{n_0}}$ of $\mathcal{S}_{x_1, \dots, x_{n_0}}$ into the generator set $\mathcal{E}_{x_1, \dots, x_{n_0}, y}$ of $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$. In particular, one has*

(3.3.4)

$$\mathcal{E}_{x_1, \dots, x_{n_0}, y} = \alpha(\mathcal{E}_{x_1, \dots, x_{n_0}}) \sqcup \{\varepsilon_{n_0}^{x_1, \dots, x_{n_0}} \varepsilon_1^y\}.$$

PROOF: By the very definition (3.3.3), the function α has its domain $\mathcal{E}_{x_1, \dots, x_{n_0}}$, whose range is contained in $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$. It is not difficult to check that α is injective. Indeed, whenever $k_1 \neq k_2$ in $\{1, \dots, n_0\}$,

$$\begin{aligned} \alpha(\varepsilon_{k_1}^{x_1, \dots, x_{n_0}}) &= \varepsilon_{k_1}^{x_1, \dots, x_{n_0}} + \varepsilon_{k_1-1}^{x_1, \dots, x_{n_0}} \varepsilon_1^y \\ &\neq \varepsilon_{k_2}^{x_1, \dots, x_{n_0}} + \varepsilon_{k_2}^{x_1, \dots, x_{n_0}} \varepsilon_1^y = \alpha(\varepsilon_{k_2}^{x_1, \dots, x_{n_0}}), \end{aligned}$$

in $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$, by (1.6).

Again by (3.3.3), the range $\alpha(\mathcal{E}_{x_1, \dots, x_{n_0}})$ of this map α is contained in the generator set $\mathcal{E}_{x_1, \dots, x_{n_0}, y}$ of $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$. Indeed, the symmetric subalgebra $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$ is generated by the elementary symmetric polynomials,

$$\varepsilon_l^{x_1, \dots, x_{n_0}, y} = \varepsilon_l(x_1, \dots, x_{n_0}, y),$$

for $l = 1, 2, \dots, n_0 + 1$, satisfying

$$\begin{aligned} \varepsilon_l^{x_1, \dots, x_{n_0}, y} &= \varepsilon_l^{x_1, \dots, x_{n_0}} + \varepsilon_{l-1}^{x_1, \dots, x_{n_0}} \varepsilon_1^y \\ &= \varepsilon_l^{x_1, \dots, x_{n_0}} + U(\varepsilon_l^{x_1, \dots, x_{n_0}}) \varepsilon_1^y = \alpha(\varepsilon_l^{x_1, \dots, x_{n_0}}), \end{aligned}$$

by (2.2), for all $l = 1, 2, \dots, n_0$. Therefore,

$$\mathcal{E}_{x_1, \dots, x_{n_0}, y} = \alpha(\mathcal{E}_{x_1, \dots, x_{n_0}}) \sqcup \{\varepsilon_{n_0+1}^{x_1, \dots, x_{n_0}, y} = \varepsilon_{n_0}^{x_1, \dots, x_{n_0}} \varepsilon_1^y\}.$$

QED

The above theorem with the relation (3.3.4) illustrates the embedding property of the generator set $\mathcal{E}_{x_1, \dots, x_{n_0}}$ of $\mathcal{S}_{x_1, \dots, x_{n_0}}$ into the generator set $\mathcal{E}_{x_1, \dots, x_{n_0}, y}$ of $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$.

Define now a linear multiplicative morphism

$$\Phi : \mathcal{S}_{x_1, \dots, x_{n_0}} \rightarrow \mathcal{S}_{x_1, \dots, x_{n_0}, y}$$

by

$$\begin{aligned} (3.3.5) \quad & \Phi \left(t_0 + \sum_{j=1}^k \sum_{(i_1, \dots, i_k) \in \{1, \dots, n_0\}^k} t_{i_1, \dots, i_k} \prod_{l=1}^k \varepsilon_{i_l}^{x_1, \dots, x_{n_0}} \right) \\ &= t_0 + \sum_{j=1}^k \sum_{(i_1, \dots, i_k) \in \{1, \dots, n_0\}^k} t_{i_1, \dots, i_k} \prod_{l=1}^k \alpha(\varepsilon_{i_l}^{x_1, \dots, x_{n_0}}), \end{aligned}$$

for all $k \in \mathbb{N}$, where $t_0, t_{i_1, \dots, i_k} \in \mathbb{C}$.

The above linear multiplicative morphism Φ of (3.3.5) is well-defined because the function α of (3.3.3) is well-defined, and it preserves the generators $\mathcal{E}_{x_1, \dots, x_{n_0}}$ injectively, by (3.3.4), into $\mathcal{E}_{x_1, \dots, x_{n_0}, y}$ of $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$.

Corollary 3.4 *The symmetric subalgebra $\mathcal{S}_{x_1, \dots, x_{n_0}}$ is algebra-monomorphic to the symmetric subalgebra $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$, for fixed variables x_1, \dots, x_{n_0} , i.e.,*
(3.3.6)

$$\mathcal{S}_{x_1, \dots, x_{n_0}} \xrightarrow{\text{Alg}} \mathcal{S}_{x_1, \dots, x_{n_0}, y},$$

where “ $\xrightarrow{\text{Alg}}$ ” means “being embedded in.”

PROOF: By the algebra-monomorphism Φ of (3.3.5), $\mathcal{S}_{x_1, \dots, x_{n_0}}$ is algebra-monomorphic to $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$, equivalently, $\mathcal{S}_{x_1, \dots, x_{n_0}}$ is naturally embedded in $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$. QED

More precise to (3.3.6), we obtain the following structure theorem.

Theorem 3.5 Let Φ be the algebra-monomorphism (or the embedding) (3.3.5) of $\mathcal{S}_{x_1, \dots, x_{n_0}}$ into $\mathcal{S}_{x_1, \dots, x_{n_0}, y}$. Then
(3.3.7)

$$\mathcal{S}_{x_1, \dots, x_{n_0}, y} \stackrel{Alg}{=} \Phi(\mathcal{S}_{x_1, \dots, x_{n_0}}) \oplus \mathbb{C}[\{\varepsilon_{n_0}^{x_1, \dots, x_{n_0}} \varepsilon_1^y\}],$$

where $\mathbb{C}[X]$ mean the algebras generated by sets X , and \oplus means the (pure-algebraic) direct product on algebras.

PROOF: Note that

$$\begin{aligned} \mathcal{S}_{x_1, \dots, x_{n_0}, y} &= \mathbb{C}[\mathcal{E}_{x_1, \dots, x_{n_0}, y}] \\ \text{by (1.7)} \quad &= \mathbb{C}[\alpha(\mathcal{E}_{x_1, \dots, x_{n_0}}) \sqcup \{\varepsilon_{n_0}^{x_1, \dots, x_{n_0}} \varepsilon_1^y\}] \\ \text{by (3.3.4)} \quad &= \mathbb{C}[\mathbb{C}[\alpha(\mathcal{E}_{x_1, \dots, x_{n_0}})] \sqcup \mathbb{C}[\varepsilon_{n_0}^{x_1, \dots, x_{n_0}} \varepsilon_1^y]] \\ \text{by construction} \quad &= \mathbb{C}[\Phi(\mathcal{S}_{x_1, \dots, x_{n_0}})] \oplus \mathbb{C}[\varepsilon_{n_0}^{x_1, \dots, x_{n_0}} \varepsilon_1^y] \\ \text{by (3.3.5)} \quad &= \Phi(\mathcal{S}_{x_1, \dots, x_{n_0}}) \oplus \mathbb{C}[\varepsilon_{n_0}^{x_1, \dots, x_{n_0}} \varepsilon_1^y], \end{aligned}$$

since Φ is an embedding. Therefore, the isomorphism theorem (3.3.7) is obtained. QED

By (3.3.7) and (2.4), we obtain the following theorem, too.

Theorem 3.6 Let $n \in \mathbb{N}$, and let $\mathcal{S}_{x_1, \dots, x_{n+1}}$ be the symmetric algebra in $\{x_1, \dots, x_n, x_{n+1}\}$. Then
(3.3.8)

$$\mathcal{S}_{x_1, \dots, x_{n+1}} \stackrel{Alg}{=} \Phi(\mathcal{S}_{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}}) \oplus \mathbb{C}[\varepsilon_n^{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n} \varepsilon_1^{x_j}],$$

where Φ is in the sense of (3.3.5).

PROOF: The proof of the structure theorem (3.3.8) is done by that of (3.3.7) in terms of (2.4). QED

We finish this section with the following example.

Example 3.1 Let $\mathcal{S}_{x_1, x_2, x_3}$ be the symmetric subalgebra in $\{x_1, x_2, x_3\}$, with its generator set

$$\mathcal{E}_{x_1, x_2, x_3} = \{\varepsilon_1^{x_1, x_2, x_3}, \varepsilon_2^{x_1, x_2, x_3}, \varepsilon_3^{x_1, x_2, x_3}\},$$

where

$$\begin{aligned} \varepsilon_1^{x_1, x_2, x_3} &= x_1 + x_2 + x_3, \\ \varepsilon_2^{x_1, x_2, x_3} &= x_1 x_2 + x_1 x_3 + x_2 x_3, \end{aligned}$$

and

$$\varepsilon_3^{x_1, x_2, x_3} = x_1 x_2 x_3.$$

For the injective map α of (3.3.3), we have

$$\begin{aligned}\alpha(\varepsilon_1^{x_1, x_2, x_3}) &= \varepsilon_1^{x_1, x_2, x_3} + \varepsilon_0^{x_1, x_2, x_3} \varepsilon_1^y = \varepsilon_1^{x_1, x_2, x_3, y} \\ \alpha(\varepsilon_2^{x_1, x_2, x_3}) &= \varepsilon_2^{x_1, x_2, x_3} + \varepsilon_1^{x_1, x_2, x_3} \varepsilon_1^y = \varepsilon_2^{x_1, x_2, x_3, y},\end{aligned}$$

and

$$\alpha(\varepsilon_3^{x_1, x_2, x_3}) = \varepsilon_3^{x_1, x_2, x_3} + \varepsilon_2^{x_1, x_2, x_3} \varepsilon_1^y = \varepsilon_3^{x_1, x_2, x_3, y},$$

inducing

$$\alpha(\mathcal{E}_{x_1, x_2, x_3}) \sqcup \{\varepsilon_4^{x_1, x_2, x_3, y} = x_1 x_2 x_3 y\} = \mathcal{E}_{x_1, x_2, x_3, y}.$$

So,

$$\begin{aligned}\mathcal{S}_{x_1, x_2, x_3, y} &= \mathbb{C}[\alpha(\mathcal{E}_{x_1, x_2, x_3}) \sqcup \{x_1 x_2 x_3 y\}] \\ &= \Phi(\mathcal{S}_{x_1, x_2, x_3}) \oplus \mathbb{C}[x_1 x_2 x_3 y].\end{aligned}$$

4 Applications

In this section, we consider applications of (3.3.7) based on (2.2).

4.1 Zeroes of Single-Variable Polynomials

Let $f(z)$ be a degree- n single-variable \mathbb{C} -polynomial, i.e., $f \in \mathcal{A}_z = \mathbb{C}[z]$. By the fundamental theorem of algebra, $f(z)$ has its zeroes $\lambda_1, \dots, \lambda_n$ (without considering multiplicities), i.e.,

$$f(z) = \prod_{j=1}^n (z - \lambda_j).$$

For convenience, let

$$x_j = -\lambda_j, \text{ for } j = 1, \dots, n.$$

Then

(4.1.1)

$$f(z) = \prod_{j=1}^n (z + x_j) = \sum_{k=0}^n \varepsilon_k^{x_1, \dots, x_n} z^{n-k}$$

in \mathcal{A}_z , where $\varepsilon_0^{x_1, \dots, x_n} = 1$, and

(4.1.2)

$$\begin{aligned}\varepsilon_k^{x_1, \dots, x_n} &= \varepsilon_k(x_1, \dots, x_n) = (-1)^k \varepsilon_k(\lambda_1, \dots, \lambda_n) \\ &= (-1)^k \varepsilon_k^{\lambda_1, \dots, \lambda_n},\end{aligned}$$

for $k = 1, \dots, n$.

Now, suppose $f(z)$ is arbitrarily given (without knowing zeroes of $f(z)$) in \mathcal{A}_z . Then one can construct the polynomial $f_{\lambda_{n+1}}(z)$ whose zeroes are the zeroes of $f(z)$ and λ_{n+1} in \mathbb{C} .

Proposition 4.1 *Let $f(z)$ be a degree- n polynomial $\sum_{k=0}^n t_k z^{n-k}$ in \mathcal{A}_z , and let $\lambda_{n+1} \in \mathbb{C}$. Then there exists a polynomial*

$$f_{\lambda_{n+1}}(z) = \sum_{k=0}^{n+1} s_k z^{(n+1)-k} \in \mathcal{A}_z$$

whose zeroes are the zeroes of $f(z)$ and λ_{n+1} in \mathbb{C} . In particular,

$$\begin{aligned} s_0 &= 1, \\ s_k &= t_k - t_{k-1} \lambda_{n+1}, \end{aligned}$$

for all $k = 1, \dots, n$, with additional identity: $t_{-1} = 0$, and

$$s_{n+1} = -t_n \lambda_{n+1},$$

in \mathbb{C} .

PROOF: Assume $f(z)$ is a degree- n polynomial in \mathcal{A}_z and $\lambda_{n+1} \in \mathbb{C}$ are arbitrarily fixed, and suppose

$$f(z) = \sum_{k=0}^n t_k z^{n-k} \text{ in } \mathcal{A}_z.$$

By the fundamental theorem of algebra, $f(z)$ has its n -zeroes $\lambda_1, \dots, \lambda_n$ (without considering the multiplicities). If we let $x_j = -\lambda_j$ in \mathbb{C} , for $j = 1, \dots, n$, then

(4.1.3)

$$f(z) = \sum_{k=0}^n t_k z^{n-k} \text{ with } t_k = \varepsilon_k^{x_1, \dots, x_n},$$

by (4.1.1) satisfying (4.1.2). So, one can construct

(4.1.4)

$$\begin{aligned} \varepsilon_0^{x_1, \dots, x_n, x_{n+1}} &= 1, \\ \varepsilon_k^{x_1, \dots, x_n, x_{n+1}} &= \alpha(\varepsilon_k^{x_1, \dots, x_n}), \text{ for } k = 1, \dots, n, \end{aligned}$$

where α is in the sense of (3.3.3), and

$$\varepsilon_{n+1}^{x_1, \dots, x_n, x_{n+1}} = \varepsilon_n^{x_1, \dots, x_n} x_{n+1} = x_1 x_2 \dots x_n x_{n+1},$$

by (3.3.4) (and by (3.3.7)), for

$$x_{n+1} = -\lambda_{n+1} \text{ in } \mathbb{C}.$$

Then, by (4.1.3), the quantities (4.1.4) are

(4.1.5)

$$\begin{aligned} s_0 &= 1, \\ s_k &= \alpha(t_k) = t_k + t_{k-1} x_{n+1}, \text{ for } k = 1, \dots, n, \end{aligned}$$

and

$$s_{n+1} = t_n x_{n+1},$$

where $x_{n+1} = -\lambda_{n+1}$.

In other words, one can construct the degree- $(n+1)$ polynomial

$$f_{\lambda_{n+1}}(z) = \sum_{k=0}^{n+1} s_k z^{(n+1)-k},$$

whose constant term s_0 and coefficients s_j satisfy (4.1.5) in \mathcal{A}_z , such that the zeroes of $f_{\lambda_{n+1}}(z)$ are the zeroes of $f(z)$ and a given \mathbb{C} -quantity λ_{n+1} . QED
The above proposition is illustrated in the following example.

Example 4.1 Let $f(z) = z^4 - 2z^2 + z + 3$ in \mathcal{A}_z , and let $i \in \mathbb{C}$. One can let

$$t_0 = 1, t_1 = -2, t_2 = 0, t_3 = 1, \text{ and } t_4 = 3,$$

in \mathbb{C} . Then, for the fixed \mathbb{C} -quantity i , we have

$$\begin{aligned} s_0 &= 1, \\ s_1 &= t_1 + t_0 i = -2 + i, \\ s_2 &= t_2 + t_1 i = 0 + (-2)i = -2i, \\ s_3 &= t_3 + t_2 i = 1 + 0i = 1, \\ s_4 &= t_4 + t_3 i = 3 + i, \end{aligned}$$

and

$$s_5 = t_4 i = 3i,$$

in \mathbb{C} , inducing a new degree-4 polynomial $f_i(z)$

$$f_i(z) = \sum_{k=0}^5 s_k z^{5-k} = z^5 + (-2 + i)z^4 - 2iz^3 + z^2 + i z + 3i.$$

in \mathcal{A}_z . Then this new degree-4 polynomial

$$z^5 + (-2 + i)z^4 - 2iz^3 + z^2 + i z + 3i$$

has its zeroes i and all zeroes of $f(z)$.

The above proposition allows us to construct a degree- $(n+1)$ polynomial $f_\lambda(z)$ whose zeroes are the zeroes of $f(z)$ and λ , even though we do not know the zeroes of a fixed degree- n polynomial $f(z)$.

4.2 JAVA Algorithm for \mathbb{C} -Quantities from Elementary Symmetric Polynomials

In this section, as an application of (3.3.7) and (2.2), we establish a computer algorithm for finding quantities from elementary symmetric polynomials. This computer algorithm is constructed by the program language, JAVA version 8.

JAVA (v.8) Program: Computing \mathbb{C} -quantities from Elementary Symmetric Polynomials.

```
import java.util.Scanner;
public class ComplexRecursiveAlgo {
public static void main(String[] args) {
Scanner in = new Scanner(System.in);
//Prompt for number of generators, n.
```

```

        System.out.print("Enter number of generating variables: ");
        int n = in.nextInt();
        System.out.println();
        //Prompt for the n known generators, vector X.
        System.out.print("Enter the values of the n generators as a white space
separating list: ");
        long[][] X = new long[n][2];
        for(int i=0; i < n; i++){
            X[i][0] = in.nextInt();
            X[i][1] = in.nextInt();
        }
        System.out.println();
        //Recursively solve for the values of the elementary symmetric functions and
store in 2D array
        long[][][] epsilon = new long[n][][2];
        for(int i=0; i < n; i++){
            epsilon[i] = new long[i+1][2];
            epsilon[0][0] = X[0];
            for(int i=1; i < n; i++){
                for(int k=0; k < epsilon[i].length; k++){
                    if(k-1 < 0)
                        epsilon[i][k] = add(epsilon[i-1][k], X[i]); //Since epsilon[i-1][k] = 1 for k < 0
                    else if(k > i-1)
                        epsilon[i][k] = multiply(epsilon[i-1][k-1], X[i]); //Since epsilon[i-1][k] = 0 for
k > i-1
                    else
                        epsilon[i][k] = add(epsilon[i-1][k], multiply(epsilon[i-1][k-1], X[i]));
                }
            }
        }
        //Provide values of epsilon to user
        System.out.print("Enter the values of n and k for the desired iteration: ");
        int N = in.nextInt() - 1;
        int K = in.nextInt() - 1;
        System.out.println();
        System.out.println("epsilon[" + (N+1) + "][" + (K+1) + "] = (" + ep-
silon[N][K][0] + "," + epsilon[N][K][1] + ")");
    }
    public static long[] add(long[] z1, long[] z2){
        long[] w = new long[2];
        w[0] = z1[0] + z2[0];
        w[1] = z1[1] + z2[1];
        return w;
    }
    public static long[] multiply(long[] z1, long[] z2){
        long[] w = new long[2];
        w[0] = z1[0]*z2[0] - z1[1]*z2[1];

```



```

w[1] = z1[0]*z2[1] + z1[1]*z2[0];
return w;
}
}
return w;
}
}

```

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